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A CASCADE MODEL FOR NEUTRALLY BUOYANT DISPERSED TWO-PHASE HOMOGENEOUS TURBULENCE—I. MODEL FORMULATION

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Abstract—This work proposes a cascade model approach to describe two-phase turbulent flows of neutrally buoyant liquid dispersions for high dispersed phase fractions. A continuum framework is selected for the representation of the mean flow momentum and continuity equations. The cascade model of Desnyansky & Novikov is extended to describe the energy spectrum and eddy intermittency in the presence of a dispersed phase. The Reynolds stress in the mean balance equation is related to the cascade variables by a spectral eddy viscosity model. A population balance is formulated for each eddy size. This approach permits the specification of the immediate environment of the drop processes modeled in the population balance. Specific drop–eddy events such as grazing collisions, drop entrapment and eddy shattering are suggested and their effects on the turbulent energy spectrum, eddy intermittencies and drop size distributions are examined. The energy, intermittency and population balance equations modified to include the effects of drop–eddy interactions form the proposed two-phase cascade model for homogeneous, neutrally buoyant, turbulent dispersions.

Key Words: cascade model, two-phase turbulence, two-phase flow, homogeneous turbulence, turbulent dispersions

1. INTRODUCTION

1.1. *Two-phase turbulence*

The droplet processes of breakage and coalescence, and the accompanying transport phenomena involving chemical reaction, are strongly coupled with the continuous phase hydrodynamics. The difficulties in making accurate predictions of turbulence statistics are compounded by the effect of the suspension on the flow. The complexity of these two-phase flows has given rise to various modeling approaches which include one-way coupling and two-way coupling models.

One-way coupling assumes that the continuous phase turbulence is unaffected by the droplets as presented by Shuen *et al.* (1983), Peskin & Kau (1979), Smith (1985) and Yuu *et al.* (1978). Although one-way coupling models are applicable under certain asymptotic conditions, the present study concerns flows with large phase fractions, for which these models are inadequate.

Formulations which account for the mutual coupling between the continuous and dispersed phases are known as two-way coupling models. When dealing with densely loaded flows, a model with this level of sophistication is usually necessary. These formulations can be classified into two categories: (1) tracking models and (2) continuum approaches. Earlier two-way coupling tracking models reported include the PSI-CELL (Crowe 1980; Crowe *et al.* 1977; Crowe & Pratt 1972; Boysan *et al.* 1982; Gosman & Ioannides 1981). Various extensions to these models allow ease of computations to track large numbers of particles (Dukowicz 1980; Kuo 1981; Kuo & Bracco 1982; O'Rourke 1981). As an alternative to the tracking methods, some investigators choose to treat multiphase mixtures as overlapping continua (Truesdell & Toupin 1960; Eringen 1976). Equations for a single "average material" with internal structure are formulated by Bedford & Drumheller

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(1983) and microstructural parameters are introduced to complete the description. A complete treatment is presented by Dobran (1985). A multiphase mixture can also be modeled by the treatment of each phase as a distinct fluid (Dopazo 1977; Chen 1983). A summary of two-fluid models is reported by Drew (1983). Subsequently, two-phase, k–ϵ, models have been developed to effect closure (Chen 1983; Chen & Wood 1985; Chen & Wood 1986; Elghobashi & Abou-Arab 1983).

Rietema & van den Akker (1983) use the control volume concept to develop generalized mean equations for the two-fluid approach. The momentum equations for the two phases are:

$$\phi\rho_c \frac{\mathrm{d}\mathbf{U}_c}{\mathrm{d}t} = \phi\nabla\cdot T_c + \nabla\cdot R_c + \phi\rho_c\mathbf{b}_c + \mathbf{f}_c \qquad [1]$$

$$(1-\phi)\rho_d \frac{\mathrm{d}\mathbf{U}_d}{\mathrm{d}t} = (1-\phi)\nabla\cdot T_d + \nabla\cdot R_d + (1-\phi)\rho_d\mathbf{b}_d + \mathbf{f}_d \qquad [2]$$

where ϕ is the volumetric phase fraction, ρ is the density, $\mathbf{U}$ is the mean velocity vector, T is the viscous stress tensor, R is the Reynolds stress tensor, $\mathbf{b}$ is the body force vector, $\mathbf{f}$ the interphase interaction force vector and the subscripts c and d refer to the continuous and dispersed phase. From left to right, the individual terms in [1] and [2] represent inertia, the viscous stress, the Reynolds stress, the body force and the interphase interaction force, respectively. Models for the Reynolds stress and the interphase interaction force must be devised in order to complete the description.

In summary, one-way coupling models are inadequate for dense dispersions; tracking models are intractable for large numbers of particles; while one-fluid and two-fluid models require adequate constitutive relations. In this study, the primary motivation is the modeling of dense liquid–liquid systems of many drops. Individual particle trajectories are not considered important. Hence, one-fluid and two-fluid approaches seem most suitable. Often, in agitated liquid–liquid dispersions, the system is nearly neutrally buoyant, and hence, the mean relative velocity between the two phases is small. It is then unnecessary to solve the particle momentum equation. In addition, the interphase interaction force, $\mathbf{f}_c$, in [1] can be neglected. In this case, the one-fluid and two-fluid models take on a very similar form, and only the Reynolds stress term needs to be modeled.

1.2. *Turbulence modeling*

In order to complete the multifluid representation, a suitable framework for the determination of the Reynolds stress must be established. In the k–ϵ model, the turbulent eddy viscosity is defined by:

$$\nu_t = C_\mu \frac{k^2}{\epsilon} \qquad [3]$$

where C_μ is a constant, k is the kinetic energy and ϵ is the turbulent energy dissipation rate. Two-equation multiphase models such as those of Chen (1983) and Elghobashi & Abou-Arab (1983) are generally unsuitable for dense dispersions because accurate modeling of the numerous cross-correlation terms, many of which are negligible in dilute dispersions, is difficult. Our approach considers the cascade model for homogeneous turbulence by Desnyansky & Novikov (1974) which exhibits the potential for the calculation of corrected values of k and ϵ. The advantage of this formulation is that no additional equations are needed for the energy dissipation rate. The eddy viscosity is then calculated on the basis of the classical model of Heisenberg (see Hinze 1975).

The energy cascade model expresses the spectral transfer of energy from large to small scales in discrete form. $u_i^2/2$ represents the average kinetic energy per unit mass of eddies of size $1/k_i$, and the conservation equations (Desnyansky & Novikov 1974), in the absence of external forcing, are

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \underbrace{\alpha k_i[(u_{i-1}^2 - 2u_{i+1}u_i)}_{\text{(I)}} - \underbrace{2^{1/3}C(u_{i-1}u_i - 2u_{i+1}^2)]}_{\text{(II)}} - \underbrace{\nu k_i^2 u_i}_{\text{(III)}} \qquad [4]$$

where u_i is the instantaneous fluctuating component of velocity of the ith eddy, α is a proportionality constant, k_i is the wave number of the ith eddy, C is the reverse spectral transfer coefficient and ν is the kinematic viscosity. Here (I) represents direct spectral transfer, (II) represents reverse spectral transfer and (III) represents viscous dissipation. The constant C measures the relative importance of reverse to direct transfer and has been assigned a value of 0.4 for decaying homogeneous turbulence in agreement with the range recommended by Bell & Nelkin (1978). In non-homogeneous flows and in the presence of a second phase, [4] must be modified to account for the spatial changes in the energy spectrum.

The advantages of the cascade model over conventional methods are several. The unique feature is that the entire energy spectrum is considered. A framework is provided within which drop–drop interactions can be examined while accounting for the local turbulent environment and not limiting the scale of interactions to properties of "typical" eddy sizes such as the inertial or energy-containing eddies. A second feature is that interphase interactions can be quantified by examination of the fundamental physical processes which occur on drop–eddy contact. There is no need for the interpretation of cross-correlation terms which appear in two-phase (k–ϵ) models (Chen 1983; Elghobashi & Abou-Arab 1983). A third feature is that dense dispersions can be described. For these reasons, a two-phase cascade model is proposed and developed in section 2.

1.3. Intermittency

The number of n eddies to be used in the two-phase cascade model is determined from the intermittency factor, β_n, which is defined as the fraction of space occupied by active n eddies (Lewalle *et al.* 1987), in conjunction with the eddy size. By this definition, the energy density $(v_n^2/2)$ is related to β_n and the cascade characteristic velocity, u_n, by

$$v_n^2 = u_n^2/\beta_n. \qquad [5]$$

Here v_n is a measure of the absolute velocities prevalent at wavenumber k_n, and is the relevant velocity parameter which describes the kinetic energy of collision of entrained objects. In order to complete the formulation, an expression for β_n suitable for multiphase flows is required.

The intermittency of turbulence in single-phase flows has been examined in several different studies (Frisch *et al.* 1978; Fujisaka & Mori 1979; Hentschel & Procaccia 1982; Mandelbrot 1976). The model of Frisch *et al.* (1978) expresses the relationship between β_n and β_{n-1} at steady state, as follows:

$$\beta_n = \beta_{n-1} 2^{D-3}. \qquad [6]$$

The similarity exponent, D, is determined experimentally. All studies agree on a value of 2.6 ± 0.1 for D. Lewalle *et al.* (1987) postulated that, in homogeneous single-phase unsteady flows, the intermittency spectrum tends to the self-similar distribution described by [6], with a time scale which is related to the local turbulence parameters, v_n and k_n. This assumption yields the following model equation for the transient intermittency:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log \beta_n) = -v_n k_n (\log \beta_n - \log(\beta_{n-1} 2^{D-3})). \qquad [7]$$

Equation [7] is unlikely to hold when droplets interfere with the normal distribution of eddies. Therefore, additional terms to account for droplets must be introduced. These modifications are discussed later in this paper. Equations [4] and [7] form the foundation for the two-phase cascade model.

1.4. Drop population balance

In addition to the eddy population, it is necessary to follow the evolution of the drop population. If the droplet size is the only internal coordinate, the general form of the drop population balance for physically equilibrated, isothermal systems (Smith 1985) is

$$\frac{\partial}{\partial t}\{N(t)f(\mathbf{x}, r_\mathrm{d}, t)\} + \frac{\partial}{\partial x_i}\{u_i(\mathbf{x}, t)N(t)f(\mathbf{x}, r_\mathrm{d}, t)\} = N(t)h(\mathbf{x}, r_\mathrm{d}, t) \qquad [8]$$

$N(t)$ represents the total number of drops in the system, $f(\mathbf{x}, r_d, t)$ is the transient probability density function, $\mathbf{x}$ is the vector of external coordinates, r_d is the drop radius, $u_i(\mathbf{x}, t)$ is the drop velocity vector and $h(\mathbf{x}, r_d, t)$ represents all source and sink terms for droplets in the control volume.

Conventionally (Valentas & Amundson 1966), droplet breakup is described by three functions: (1) $\nu(v)$, the average number of droplets formed from a breaking drop of volume v; (2) $\beta(v', v)$, the daughter droplet size probability density function produced by a parent of volume v; and (3) $g(v)$, the frequency of breakage of a drop of volume v. Drop coalescence is expressed in terms of two functions: (1) $h(v, v')$, the frequency collisions between drops of sizes v and v' and (2) $\lambda(v, v')$, the coalescence efficiency of collisions between drops of sizes v and v'. Expressed in terms of these functions, the spatially homogeneous population balance equation with no feed and exit events is

$$\frac{\partial N(v,t)}{\partial t} = -g(v)N(v,t) + \int_v^{v_{\max}} \nu(v')\beta(v,v')g(v')N(v',t)\,dv' + \int_0^{v/2} \lambda(v-v',v')h(v-v',v') \times N(v-v',t)N(v',t)\,dv' - N(v,t)\int_0^{(v_{\max}-v)} \lambda(v,v')h(v,v')N(v',t)\,dv'. \quad [9]$$

$N(v, t)$ is henceforth referred to as the number density, and is the product of $N(t)$ and $f(r_d, t)$.

Several studies (Clark 1988a, b; Lagisetty *et al.* 1986; Koshy *et al.* 1988) have examined drop breakup in turbulent flow. Clark (1988a) developed a dynamic model for the breakage of droplets larger than the Kolmogorov microscale.

Lagisetty *et al.* (1986) proposes a model for drop breakage in which the deformation of the drop is represented by a Voigt element. Koshi *et al.* (1988) extended this framework to account for surfactants by the inclusion of an extra stress in the stress balance.

The approach used in this study is based on the models of Coulaloglou & Tavlarides (1977). The breakage frequency of drops is calculated as a product of the fraction of drops breaking, and the breakage time. The former is modeled by determination of the fraction of drops with turbulent kinetic energy greater than the drop surface energy. The latter is determined by analogy with the separation of two lumps of fluid in a turbulent flow. The relative motion of the potential daughter droplets is assumed similar to that of the two lumps. The breakage frequency is therefore expressed as (Bapat & Tavlarides 1985):

$$g(v) = \frac{C_1 \epsilon^{1/3}}{(1+\phi)v^{2/9}} \exp\left[-\frac{C_2 \sigma (1+\phi)^2}{\rho_d \epsilon^{2/3} v^{5/9}}\right] \quad [10]$$

where C_1 and C_2 are constants, v is the drop volume and σ is the interfacial tension.

The daughter droplet distribution function is assumed to be Gaussian (Coulaloglou & Tavlarides 1977), and only binary breakage is considered. Therefore

$$\beta(v, v') = \frac{2.4}{v'} \exp\left[-4.5\frac{(2v - v')^2}{(v')^2}\right], \quad [11]$$

and

$$\nu(v) = 2. \quad [12]$$

Droplet collision is modeled by assuming that the process is similar to the collision of gas molecules, as in the kinetic theory of gases. Then the collision frequency is expressed as (Coulaloglou & Tavlarides 1977).

$$h(v_1, v_2) = C_3(v_1^{1/3} + v_2^{1/3})(\bar{u}_1^2 + \bar{u}_2^2)^{1/2}/(1+\phi) \quad [13]$$

where C_3 is a constant. In the model of Coulaloglou & Tavlarides (1977), $\bar{u}_1^2$ and $\bar{u}_2^2$ are the mean squared velocities of drops with volumes v_1 and v_2 and are given in terms of the average turbulence properties as

$$\bar{u}_j^2 = k_1 \epsilon^{2/3} v_j^{2/3} \quad [14]$$

where k_1 is a constant. The coalescence efficiency of droplet collisions is modeled by estimating the fraction of binary drop collisions which result in coalescence. By assuming that the rate determining step is the time needed for the intervening film between the drops to drain to a critical thickness and that the compressive force along the lines of centers of the two drops, which cause the film drainage, is a result of the isotropic turbulent motion of eddies, the coalescence efficiency can be written as

$$\lambda(v_1, v_2) = \exp\left[\frac{C_4 \mu_c \rho_c \epsilon}{\sigma^2(1+\phi)^3}\left[\frac{v_1^{1/3} v_2^{1/3}}{v_1^{1/3} + v_2^{1/3}}\right]^4\right] \quad [15]$$

where μ_c is the viscosity of the continuous phase and C_4 is a constant. These functions can be used to describe droplet–eddy interactions as shown below.

2. THE TWO-PHASE CASCADE MODEL

In order to describe the flow of two phases, [4], [7] and [9] must be corrected to account for interphase interactions. It is presumed that the structure of the terms already present in the equations is not changed by these interactions although their numerical value will be different. The presence of the dispersion is reflected through additional dissipative terms introduced into the energy budget, and via source and sink terms added to the intermittency and population balance equations. The general form of the two-phase cascade model is therefore

$$\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) = \alpha k_n[u_{n-1}^2 - 2u_{n+1}u_n - 2^{1/3}C(u_{n-1}u_n - 2u_{n+1}^2)]q - \nu k_n^2 u_n + \{\text{drop–eddy interactions}) \quad [16]$$

$$\frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n v_n k_n[\log \beta_n - \log(\beta_{n-1}2^{D-3})] + \{\text{drop–eddy interactions}\} \quad [17]$$

where k_n are discretized wave numbers.

Since the mutual interactions between eddies and drops are likely to depend on the local environment, a more accurate description can be expected if a separate homogeneous drop population balance is written for each eddy size rather than for the cumulative statistics over all sizes. The advantage of the approach is two-fold. First, the drop population is distributed over the entire population of eddies, and its evolution therefore reflects interactions with the whole spectrum. Second, the drop size distribution and phase fraction could differ substantially between eddy sizes. The cascade framework permits the description of the effect of these quantities on the local energy spectrum and drop dynamics.

Accordingly, [9] is modified to represent the local population balance. The subscript 'n' is affixed to the number density $N(v, t)$, the maximum drop volume $v_{\max}$, and the droplet functions $g(v)$, $\beta(v, v')$, $v(v)$, $\lambda(v, v')$ and $h(v, v')$, to indicate that they represent quantities or functions local to eddy size n.

$$\begin{aligned}\frac{\partial N_n}{\partial t}(v,t) &= -g_n(v)N_n(v,t) + \int_v^{(v_{\max})_n} v_n(v')\beta_n(v,v')g_n(v')N_n(v',t)\,\mathrm{d}v' \\ &+ \int_0^{v/2} \lambda_n(v-v',v')h_n(v-v',v')N_n(v-v',t)N_n(v',t)\,\mathrm{d}v' \\ &- N_n(v,t)\int_0^{(v_{\max})_n - v} \lambda_n(v,v')h_n(v,v')N_n(v',t)\,\mathrm{d}v' + \{\text{drop–eddy interactions}\}. \end{aligned} \quad [18]$$

The initial conditions for [16], [17] and [18] are prescribed as

$$u_n(0) = u_n^0; \quad \beta_n(0) = \beta_n^0; \quad N_n(v,0) = N_n^0(v); \quad n = 1, \ldots, N. \quad [19]$$

The {drop–eddy interaction} terms in [16]–[18] will be constructed in the following sections.

2.1. Turbulence structure

In the derivation of the energy cascade equations, it was observed that interactions between length scales were strongest when the associated time scales were comparable. This concept laid the foundation for the assumption that the energy equation associated with the length scale "n" is coupled only to the energy equations of its "nearest neighbors", scales "$n - 1$" and "$n + 1$". In other words, "n eddies" were presumed to exchange energy only with "($n + 1$) eddies", "($n - 1$) eddies", and other "n eddies", because their time scales matched. Thus, a snapshot of a collection of eddies would show "($n + 1$) eddies" embedded primarily in "n eddies", "n eddies" in "($n - 1$) eddies", and so on. Neglecting all but nearest neighbor interactions is then tantamount to assuming that eddies of an arbitrary size class "n" embed ONLY "($n + 1$) eddies". Of course, the largest eddies must then fill the flow, and the smallest must contain no fluctuations at all. The idealized picture depicted in figure 1 was employed by Lewalle *et al.* (1987). It may be observed that an "($n - 1$) eddy" may include several fluctuations, all of which fall within the size class "n".

The idealization of the eddy structure made in the preceding paragraph is merely an extension of the assumption of nearest neighbor interactions. It is also consistent with the picture of the eddy population to presume that a drop in an "n eddy" encounters only other species (drops or eddies) in the same eddy. Since all embedded eddies are of size class "$n + 1$", the possible collision combinations are reduced substantially. The collision parameters are well defined, and the problem is transformed into a more tractable one. The formulation also specifies the environment of drop–drop encounters in "n eddies" to be the prevailing state of the "n eddy".

Although the hypothesis of successively contained eddies appears attractive, it must be employed with caution. For example, in the vicinity of a wall, the large eddy length scale changes rapidly as the wall is approached. The dispersion could then exist outside of the large eddies, thus violating the model assumption that eddies fill the space. Under these conditions, the hypothesis of successively contained eddies must surely break down. In fact, it is possible to construct a variety of configurations in which a more elaborate eddy structure is needed for a realistic description.

2.2. *Collision frequencies*

In order to quantify interphase and intraphase interactions, the drop–eddy and drop–drop collision frequencies in a specified local turbulence environment must be estimated. Coulaloglou & Tavlarides (1977) assume that the mechanism of collision in a locally isotropic flow field resembles that of molecules in the kinetic theory of gases. The collision rate is then calculated by estimating the mean square fluctuating velocities of drops in terms of the drop size and mean rate of turbulent energy dissipation. A similar model is constructed in the two-phase cascade.

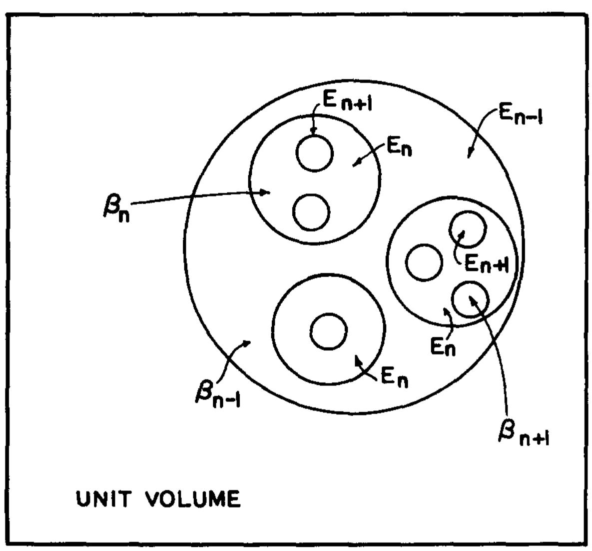

Figure 1. Idealized eddy structure.

If N_E and N_D represent the number of objects of each type that partake in the collision, the number of collisions experienced per unit time in volume V assuming random collisions is given by

$$Z_E = \frac{\pi(r_d + R_e)^2}{V} v_r N_E N_D, \qquad [20]$$

where r_d and R_e are the radii of the spherical colliding objects, and v_r is their relative velocity. The number of drops in the size range $[v, v + dv]$ per unit volume of dispersion is $N(v)\,dv$, and the number of eddies of size n with radius of l_n is $(\beta_n/((4\pi/3)l_n^3))$. Therefore,

$$Z_E = \pi(r_d(v) + l_n)^2 v_r N(v)\left(\frac{3\beta_n}{4\pi l_n^3}\right) dv, \qquad [21]$$

and the collision frequency is

$$f(n, v) = \pi(r_d(v) + l_n)^2 v_r \left(\frac{3\beta_n}{4\pi l_n^3}\right). \qquad [22]$$

In agreement with the simplified turbulence structure discussed in section 2.1, all species residing within a parent of size n move randomly with a characteristic speed v_n. The available volume V in the parent eddy per unit volume of the dispersion is taken to be β_n and the average relative velocity v_r is found by integrating over all possible collision angles. Hence,

$$v_r = \int_0^\pi 2v_n \sin\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\sin\theta\right) d\theta = \frac{4v_n}{3}. \qquad [23]$$

In the successively contained scheme, the collision frequency is therefore

$$f(n+1, v) = \frac{K_4 v_n (r_d(v) + l_{n+1})^2 \beta_{n+1}}{l_{n+1}^3 \beta_n} \qquad [24]$$

where K_4 is a constant.

As pointed out by Lewalle *et al.* (1987), the random collision model is unrealistic in the case of sub-microscale droplets, which are exposed to laminar motion, and hardly ever collide. To account for this case, the authors derive an expression for the collision frequency assuming a homogeneous distribution of droplets in a uniform rate of strain. A collision is established if the trajectory of one of the objects is such that contact occurs within the expected life of the embedding eddy. The collision velocity is taken to be the instantaneous value at the point of impact in the undisturbed flow. The collision frequency per drop unit volume of the embedding eddy is given by (Lewalle *et al.* 1987):

$$f(n+1, v) = 1.22 \frac{v_n}{l_n}(r_d(v) + l_{n+1})^3 n_e \qquad [25]$$

where n_e represents the number of $(n + 1)$ eddies embedded in a unit volume of eddies of size n. The collision velocity is

$$v_r = \frac{v_n}{2l_n}(r_d(v) + l_{n+1}). \qquad [26]$$

It is instructive to compare the relative merits of the ordered and random collision models. In drop–eddy collisions, one of the objects, the eddy is always at least as large as the Kolmogoroff microscale. Under these conditions, the random collision model and the uniform rate of strain model give similar results for collision frequencies. The random collision model is preferred because it is consistent with the coalescence models of Coulaloglou & Tavlarides (1977). Also, the mathematical treatment of drop trajectory information in random drop-eddy collisions is more tractable. An analysis of drop motion is presented below, based on the random collision assumption.

For collisions between two drops, both much smaller than the embedding eddy, the uniform rate of strain seems to be a more accurate representation and is adopted here. These collisions are assumed to be partially inelastic, and their contributions to the energy budget are considered.

2.3. *Interactions*

The objective of this section is to develop a reasonable description of the interaction terms which appear in [16]–[18] and to produce a working model which adequately describes the mutual effects between the dispersion and turbulence. From the turbulence structure modeled in section 2.1, it follows that a drop trapped in an "n eddy" could behave in one of the following ways:

(a) It may encounter an entrained "$(n + 1)$ eddy", and perhaps, enter its confines.
(b) It may leave the region of influence of the embedding "n eddy" to enter a larger "$(n - 1)$ eddy" by mechanisms which are examined later in this paper.
(c) It may undergo breakage or coalescence, and hence, alter its identity.

In (a) and (b), the drop–eddy encounter provides a change in environment for the droplet, for at least a short interval of time. During this period, the relative velocity between the drop and its surroundings could be substantial. All the while, the new environment retards the droplet motion, diminishing the relative velocity and dissipating energy in the process. This energy must stem from some point in the cascade itself, thus producing a damping of turbulence. In the ensuing analysis, it is presumed that this drop–eddy dissipation phenomenon is responsible for the weakening of the energy spectrum. The system can be viewed as a series of dissipative collision-type "events", each of brief duration, separated by a quiescent period in which interphase energy dissipation is negligible. The interaction time is assumed to be small compared with the time between events and the eddy turnover time. This permits each event to proceed uninterrupted by other rapid changes in the system.

Interphase interactions. As observed earlier, in a system consisting of drops and eddies, three forms of interaction exist—eddy–eddy, drop–drop and eddy–drop. The former involves the turbulent continuum only, and is, presumably, accounted for by the single-phase cascade terms which appear in [16] and [17]. Drop–drop encounters can be classified into coalescence and breakage events. An adaptation of the coalescence models of Coulaloglou & Tavlarides (1977) as described earlier is used here. Breakage, on the other hand, may occur due to local shear, or due to pressure fluctuations in the droplet surface. Both phenomena produce the same result. The latter, however, is presumed to occur on drop–eddy contact, and is therefore treated as such.

Of the three interaction forms mentioned above, eddy–drop phenomena are least understood. For this reason, special attention has been paid to them. It is presumed that the energy losses sustained do not affect single-phase cascade (eddy–eddy) terms at all. Instead, they appear only in the form of additive terms in the energy budget. This approach has a decoupling effect on the various interaction types, whose contributions can hence be computed separately. In the following section, five such interaction types are postulated, their influence on the cascade equations is discussed qualitatively, and the resulting interaction terms are derived.

Drop–eddy interactions. The following set of five drop–eddy interactions which can account for partially inelastic collisions is postulated:

(1) drop breakage
(2) grazing collisions, or drop exit
(3) drop entrapment
(4) eddy shattering and
(5) eddy turnover or dumping.

Each of these interactions is considered in succession.

(1) Drop breakage. The drop breakage outcome of drop–"n eddy" impact is assumed to be caused by pressure fluctuations on the droplet surface. The change in surface energy due to the breakage of a drop of volume v into daughters of volume v' and $(v - v')$ is given by:

$$E_S = (36\pi)^{1/3}\sigma[v^{2/3} - v'^{2/3} - (v - v')^{2/3}]. \qquad [27]$$

Then the number of drops in "n eddies" in the size range $[v, v + dv]$ which break per second per unit volume of dispersion, to produce a daughter in the size range $[v', (v' + dv')]$ is given by

$$N_n(v)\,dv g_n(v)[\beta_n(v', v)\,dv'], \qquad [28]$$

where $g_n(v)$ is the breakage frequency of size v drops in "n eddies" and $\beta_n(v', v)\,dv'$ is the fraction of breaking drops of size v which produce a daughter in the size range $[v', (v' + dv')]$. Therefore, the energy per unit volume per unit time extracted from the flow during breakage events in "n eddies" is

$$(\Delta E_B)_n = \int_{(v_{min})_n}^{(v_{max})_n} \int_{(v_{min})_n}^{v/2} E_n \, dv' \, dv \qquad [29]$$

where

$$E_n = (36\pi)^{1/3} \sigma N_n(v) g_n(v) \beta_n(v, v')(v^{2/3} - v'^{2/3} - (v - v')^{2/3}) \qquad [30]$$

and $E_n\,dv'\,dv$ is the differential surface energy change per unit mass due to drop breakage in "n eddies" and $(v_{min})_n$ is the minimum volume of drops in "n eddy".

Drop breakage does not affect the eddy intermittency distribution, but it alters the number density of the parent "n eddy". The sink term due to drop breakage in the equation is given by

$$(B_1)_n = -g_n(v) N_n(v). \qquad [31]$$

The source term due to drop breakage is

$$(B_2)_n = \int_v^{(v_{max})_n} \nu_n(v') \beta_n(v, v') g_n(v') N_n(v') \, dv' \qquad [32]$$

where $\nu_n(v')$ is the average number of daughter droplets formed during breakage of a drop of volume v'.

Equations [29]–[32] represent the contribution of drop breakage to the two-phase cascade model. Breakage functions $g_n(v)$, $\beta_n(v, v')$ and ν_n are given by [10]–[12].

(2–4) Exit, entrapment and eddy shattering. These three collision outcomes are closely related and are best discussed simultaneously. If drop breakage occurs during a drop–"n eddy" collision, it is assumed to take place on impact. Consequently, if this event does not occur, the drop enters the confines of the eddy. It is then acted upon by a dissipative retarding force, extered by its new environment. If its entrance kinetic energy is large enough, the resistive forces are insufficient to bring it to equilibrium with its new surroundings. Instead, the drop passes through the eddy, loses some of its kinetic energy in the process and returns to its original "$(n - 1)$ eddy" environment. This occurrence is referred to as an exit event or a grazing collision. If, on the other hand, the drop is brought to rest, it remains trapped in the colliding "n eddy". This phenomenon is known as entrapment. Finally, if the drop motion disorganizes the eddy sufficiently, the eddy may lose its coherence, and hence, its identity. Presumably, the fluid mass remaining will show up as part of the parent eddy. The present model assumes that this phenomenon occurs whenever the energy dissipated during the interaction is comparable with the turbulent energy contained in the colliding "n eddy". This event-type is called eddy shattering.

Each of the three interaction types has a different effect on the cascade equations. All three events result in the dissipation of energy. Hence, the local energy budget must reflect their possible occurrence. Exit and entrapment events leave the integrity of the eddy untouched. Since the number of eddies in the dispersion remains unaffected by these events, the intermittency equations remain unchanged. Shattering on the other hand, actually destroys the colliding eddy. This effect is reflected by a sink term in the intermittency equation. Finally, the local population balances are affected only by entrapment and shattering. Exit events clearly play no part in these balances since, at the termination of the process, the drop returns to the parent environment. Entrapment, on the other hand, results in the net movement of the colliding drop from an "$(n - 1)$" parent to an "n eddy". The effect of shattering on the drop number distribution is more subtle. The destroyed eddy is known to have harbored a population of drops prior to the collision. On disintegration of their surroundings, these drops reappear in the environment of the "$(n - 1)$" parent. The net effect is their migration from an "n eddy" to an "$(n - 1)$ eddy". The number balances of both n and $(n - 1)$ eddies, therefore, reflect entrapment and shattering.

Among the collision parameters, a probability distribution of drop–eddy collision angles is considered, in order to determine the distribution of the relative velocity of collision. The relevant frequencies and energies can be calculated exactly for a spherical eddy, and permits a convenient

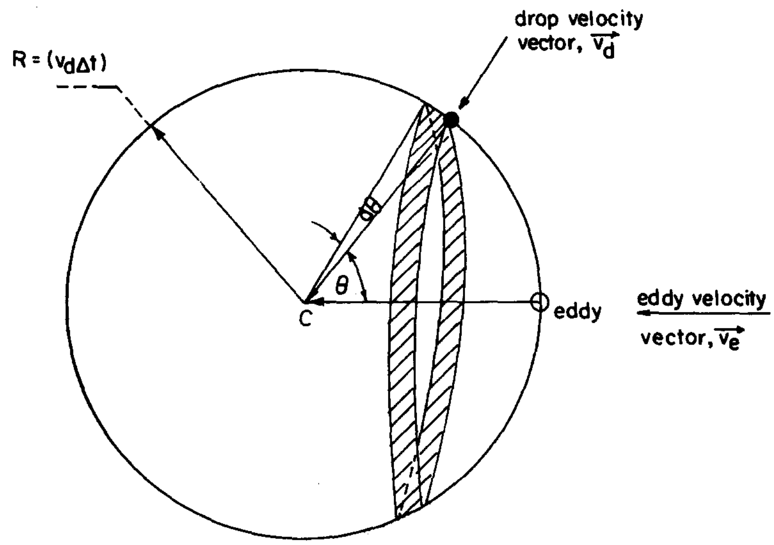

Figure 2. Probability density of drop–eddy collision angle.

mathematical description of the process. These values so calculated are presumed to be good estimates for more realistic eddy shapes characterized by a unique length scale. The probability distribution can be derived by observing the motion of the center of the drop from the reference frame of the eddy. Then the probability that the collision angle lies between θ and $(\theta + \mathrm{d}\theta)$ is proportional to the area of the shaded region in figure 2.

$$g(\theta)\,\mathrm{d}\theta = \frac{(2\pi R\sin\theta)(R\,\mathrm{d}\theta)}{4\pi R^2} = \frac{1}{2}\sin\theta\,\mathrm{d}\theta \qquad [33]$$

where R is the radius of the sphere. Since the collision occurs in an "$(n-1)$" parent, this characteristic speed is assumed to be v_{n-1}. The direction of motion of each species is a random variable with a uniform probability density. Then, the relative velocity of collision (figure 3) is given by:

$$v_r = 2v_{n-1}\sin\left(\frac{\theta}{2}\right). \qquad [34]$$

The probability that the collision radius will lie between r and $r + \mathrm{d}r$ is equal to the cross-hatched area divided by the eddy cross-section. Therefore

$$f(r)\,\mathrm{d}r = \frac{2\,\mathrm{d}r}{(R_e)_n^2} \qquad [35]$$

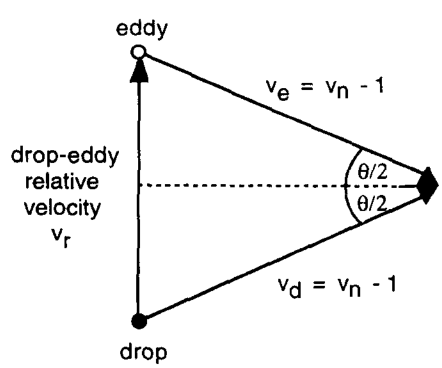

Figure 3. Drop–eddy relative velocity.

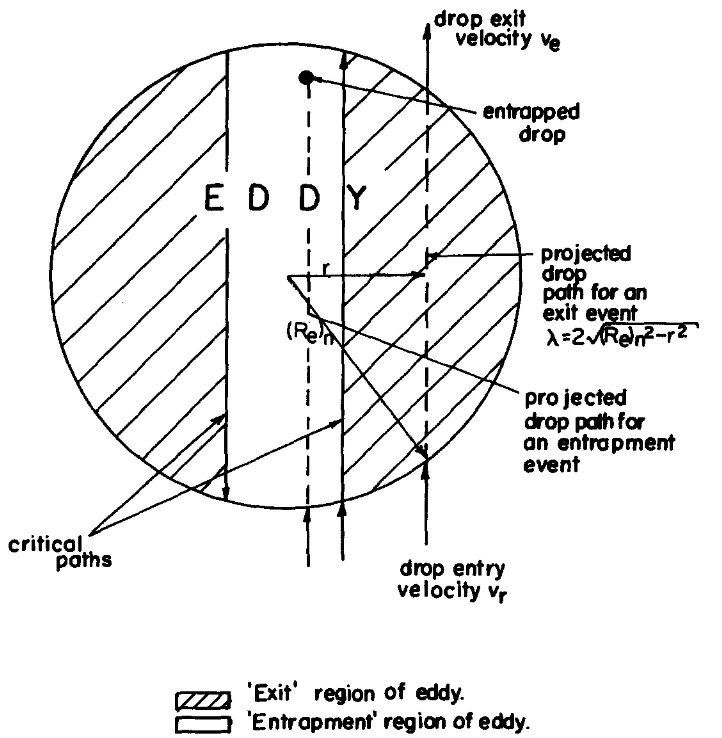

Figure 4. Drop exit and entrapment.

where $f(r)$ is the probability density function for collision radius r and $(R_e)_n$ is the radius of the "n eddy". θ uniquely determines the relative velocity, while r represents the radius of contact.

Following the collision, the drag experienced by the drop is modeled after slow motion of a solid particle in an infinite medium—Stokes Law.

The path length, λ, of the center of the drop in an exit event from an eddy (figure 4) can be related to the radius of contact r by:

$$\lambda = 2((R_e)_n^2 - r^2)^{1/2}. \quad [36]$$

It follows that if the drop is brought to rest (with respect to the eddy) within a distance λ, entrapment occurs. If not, the result is an exit event (figure 4). If, during the drop motion, the energy dissipated exceeds the eddy energy, the result is a shattering event.

The critical radius, R_c, below which the drop stays trapped is related to the maximum stopping distance, L, of a particle in Stokes flow by:

$$R_c = \begin{cases} ((R_e)_n^2 - L^2/4)^{1/2} & (R_e)_n > L/2 \\ 0, & (R_e)_n \leqslant L/2 \end{cases}. \quad [37]$$

An expression for L can be derived by integration of the particle equation of motion. In the reference frame of the continuum,

$$\frac{4\pi}{3} r_d \rho_d \frac{dv_d}{dt} = -6\pi\mu r_d v_d, \quad [38]$$

from which upon integration

$$\frac{v_d}{v_r} = e^{-(9\mu t)/(2\rho_d r_d^2)}. \quad [39]$$

where μ is the viscosity and v_d is the drop velocity. Therefore,

$$L = \int_0^L ds = \int_0^\infty v_r \exp(-9\mu t/2\rho_d r_d^2)\, dt = \frac{2\rho_d r_d^2 v_r}{9\mu}. \quad [40]$$

In general, the instantaneous distance traveled can be related to the instantaneous velocity by:

$$L(v_d) = \frac{2\rho_d r_d^2}{9\mu}(v_r - v_d). \quad [41]$$

A second critical radius of relevance, r_c, the critical collision radius which distinguishes between drop exit and eddy shattering, is that radius above which the drop path is too short to permit shattering. Clearly, when the initial kinetic energy of the drop is smaller than the eddy, energy scattering does not occur even if all the kinetic energy is dissipated. On the other hand, if the initial drop energy is large enough, and the collision radius less than r_c, the energy dissipation exceeds the eddy energy, which results in eddy destruction. Therefore, the minimum energy that must be dissipated for decimation of the eddy is given by:

$$(\Delta E)_l = \frac{\rho_m}{2}\left(\frac{4\pi(R_e)_n^3}{3}\right)v_n^2. \quad [42]$$

The kinetic energy lost by the drop as a function of drop velocity is

$$E(v_d) = \frac{\rho_d V_d}{2}(v_r^2 - v_d^2). \quad [43]$$

At $r = r_c$, the path length is just sufficient such that $(\Delta E)_l = E(v_d)$. Accordingly,

$$\frac{\rho_d V_d}{2}(v_r^2 - v_e^2) = \frac{4\pi\rho_m(R_e)_n^3}{6}v_n^2. \quad [44]$$

The exit drop velocity, v_e, can be expressed in terms of path length from [41] as:

$$v_e = v_r - \frac{9\mu\lambda}{2\rho_d r_d^2}. \quad [45]$$

Substituting for λ from [36],

$$v_e = v_r - \frac{9\mu\sqrt{(R_e)_n^2 - r_c^2}}{\rho_d r_d^2} \quad [46]$$

Equations [44] and [46] can be solved for r_c to obtain

$$r_c = \left[(R_e)_n^2 - \frac{2v_r^2\rho_d^2 r_d^4}{81\mu^2}\left(1 - \left(1 - \frac{(R_e)_n^3\rho_m v_n^2}{v_r^2 r_d^3\rho_d}\right)^{1/2}\right) - \frac{(r_e)_n^3\rho_m v_n^2}{2v_r^2 r_d^3\rho_d}\right]^{1/2}. \quad [47]$$

Clearly, r_c does not exist, i.e. shattering is not possible, if [44] can never be satisfied for real r. In this case:

$$(\rho_d V_d)/2(v_r^2) < (2\pi/3)\rho_m(R_e)_n^3 v_n^2. \quad [48]$$

Inspection of [37], [40] and [47] indicates that both critical radii depend on v_r, which in turn depends on the collision angle θ. In other words, given a value for the random variable, θ, the collision radius, r, and the drop radius, r_d, [37], [40], [47] and [48] determine which event—exit, entrapment, or shattering—occurs. A more detailed discussion on the variation of the critical radii, and hence the probabilities of the outcomes, with the collision angle θ is presented by Jairazbhoy (1989). Figure 5 distinguishes between drop exit and eddy shattering events.

In summary, the critical quantities r_c, R_c, θ_c and θ_0 determine active ranges for the random variables r and θ, and hence, limits for their integration. Here θ_c is the critical angle above which scattering is possible and θ_0 is the angle at which R_c falls to zero. A more detailed analysis of the effects of the two random variables r and θ, and the drop radius r_d on the event outcomes is presented elsewhere (Jairazbhoy 1989).

The influence of the three event types, described above, on the Cascade equations is examined next.

Energy equation: the energy lost in a single occurrence of an exit event is given by:

$$E_E = \frac{\rho_d V_d}{2}(v_r^2 - v_e^2), \quad [49]$$

where the exit velocity, v_e, is a function of the three random variables θ, r and r_d and V_d is the drop volume. Therefore, the total energy lost by "n eddies" per unit volume of dispersion due to exit events is

$$(\Delta E_E) = \int_{(v_{min})_{n-1}}^{(v_{max})_{n-1}} \int_\theta \int_r \rho_d V_d \left(2v_{n-1}^2 \sin^2\frac{\theta}{2} - \frac{v_e^2}{2}\right) f(n, v)[1 - B_f(n, v)] \times g(\theta) f(r) N_{(n-1)}(v)\, dr\, d\theta\, dv. \quad [50]$$

where $f(n, v)$ is the collision frequency for drops between n eddies and drops of size v and $B_f(n, v)$ is the fraction of "n eddy"–drop (of volume v) collisions which result in drop breakage. Similarly, the contribution of drop entrapment to the energy equation is given by:

$$(\Delta E_T)_n = \int_{(v_{min})_{n-1}}^{(v_{max})_{n-1}} \int_\theta \int_r 2(\rho_d V_d) v_{n-1}^2 \sin^2\frac{\theta}{2} f(n, v)[1 - B_f(n, v)] \cdot N_{n-1}(v) g(\theta) f(r)\, dr\, d\theta\, dv, \quad [51]$$

where $(\Delta E_T)_n$ is the energy lost per unit volume by "n eddies" due to entrapment. The energy lost per unit volume by "n eddies" due to shattering is:

$$(\Delta E_s)_n = \int_{(v_{min})_{n-1}}^{(v_{max})_{n-1}} \int_\theta \int_r \left(\frac{\rho_m (V_e)_n v_n^2}{2}\right) f(n, v)[1 - B_f(n, v)] \cdot N_{n-1}(v) g(\theta) f(r)\, dr\, d\theta\, dv. \quad [52]$$

The limits on r and θ in the above equations must be chosen such that the result of the collision for $(v_{min})_{n-1} < v < (v_{max})_{n-1}$ is exit [50], entrapment [51] or shattering [52]—as explained by Jairazbhoy (1989). The collision frequency function $f(n, v)$ is given by [22] and the breakage function $B_f(n, v)$ is derived in Jairazbhoy (1989) and given as

$$B_f(n, v) = \exp\left\{-\frac{v_c^2}{v_n^2}\right\}. \quad [53]$$

Here v_n is the velocity of the "n-eddy", and v_c is the critical velocity of the "n-eddy" which contains enough energy to overcome the drop surface energy.

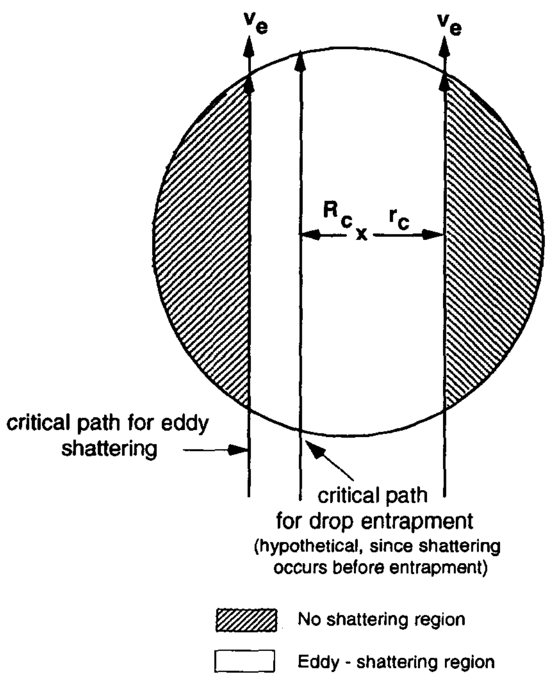

Figure 5. Eddy shattering.

Equations [50]–[52] represent the contributions of exit, entrapment and shattering to the energy equation. The right hand sides of these equations can be simplified by performing the integration over the variables r and θ analytically (Jairazbhoy 1989).

Intermittency equation: the only event type which affects the eddy intermittency is eddy shattering. The change in interittency per unit time due to such events is given by:

$$(\Delta\beta)_n = (V_e)_{n+1} \int_{(v_{\min})_n}^{(v_{\max})_n} (\Delta B_s)_{n+1}\,dv - (V_e)_n \int_{(v_{\min})_{n-1}}^{(v_{\max})_{n-1}} (\Delta B_s)_n\,dv \qquad [54]$$

where $(V_e)_n$ is the "n eddy" volume and $(\Delta B_s)_n$ represents the number of "n eddies" that shatter per unit tie per unit volume of dispersion, and is given by:

$$(\Delta B_s)_n = \int_\theta \int_r N_{n-1}(v) f(n, v)[1 - B_f(n, v)] g(\theta) f(r)\,dr\,d\theta \qquad [55]$$

where $g(\theta)$ is the probabiity density function for collision angle θ.

Drop number balance: the number of drops in "n eddies" of volume between v and $v + dv$ is affected by source and sink terms associated with eddy shattering and drop entrapment.

The net contribution of eddy shattering to the local population balance is given by:

$$F_s(v) = N_{n+1}(v) \int_{(v_{\min})_{n+1}}^{(v_{\max})_{n+1}} \int_\theta \int_v f(n+1, v')[1 - B_f(n+1, v')] N_n(v') g(\theta) f(r)\,dr\,d\theta\,dv'$$

$$- N_n(v) \int_{(v_{\min})_n}^{(v_{\max})_n} f(n, v')[1 - B_f(n, v')] N_{n-1}(v') g(\theta) f(r)\,dr\,d\theta\,dv'. \qquad [56]$$

The net contribution of entrapment to the local population balance is represented by:

$$F_T(v) = \int_\theta \int_v N_{n=1}(v) f(n, v)[1 - B_f(n, v)] g(\theta) f(r)\,dr\,d\theta$$

$$- \int_\theta \int_v N_n(v) f(n+1, v)[1 - B_f(n+1, v)] g(\theta) f(r)\,dr\,d\theta. \qquad [57]$$

Details describing the analytical evaluation of the integrals over r and θ in [50]–[57] are reported elsewhere (Jairazbhoy 1989).

(5) Eddy turnover or dumping. This interaction is the last of the five postulated eddy–drop phenomena. It is different from the others inasmuch as it does not directly involve an eddy–drop collision. Since eddies have a limited life span, it is necessary to examine what happens to drops trapped within an eddy, when the eddy dies. Clearly, these drops experience a change in environment. This clearly causes a momentary relative velocity between the two phases, which is subsequently reduced by the action of dissipative forces. This phenomenon is referred to as eddy turnover or dumping. The extent to which energy is dissipated during this process, and the ultimate location of the entrapped drops, remain to be examined.

The time scale associated with "n eddies", $t_n = l_n/u_n$ is a reasonable estimate for the local eddy turnover time. Also, something must be said about the eddy turnover process. The simplest approach would be to assume that eddies are statistical entities, that form and disappear with a characteristic frequency. When a new eddy forms as a growing instability, it seems reasonable to consider the entrained drop number density to be similar to that of the parent eddy. Care must be taken, however, to prohibit entrainment of drops larger than the maximum allowable in the daughter eddy. Conversely, the decay of an instability, or disappearance of an eddy, presumably results in the deposition of the entrained drops into the parent eddy. Assuming that these processes are entirely dissipative, the energy lost by "n eddies" per unit time per unit volume of dispersion due to dumping, $(\Delta E_D)_n$, is:

$$(\Delta E_D)_n = (u_n k_n)\left(\frac{\rho_d v_n^2}{2}\right)\left[\frac{\beta_n}{(\beta_{n-1} - \beta_n} \int_{(v_{\min})_{n+1}}^{(v_{\max})_{n+1}} v' N_{n+1}(v')\,dv' + \int_{(v_{\min})_n}^{(v_{\max})_n} v' N_n(v')\,dv'\right]. \qquad [58]$$

The net contribution of dumping to the population balance is

$$F_D(v) = u_{n+1}k_{n+1}(N_{n+1}(v) - N'_n(v) + u_n k_n(N'_{n-1}(v) - N_n(v)), \quad [59]$$

where $N'_n(v)$ is introduced to prohibit large drops from entering small eddies and is defined as

$$N'_n(v) = \begin{cases} N_n(v)\dfrac{\beta_n}{(\beta_{n-1}-\beta_n)} & v \leqslant (v_{\max})_{n+1}, \\ 0, & v > (v_{\max})_{n+1}. \end{cases} \quad [60]$$

Since eddy formation and disappearance are already accounted for by single phase terms in the intermittency equation, dumping has no additional effect.

2.4. Two-phase cascade equations

The contributions of all five interaction types are introduced into the cascade model to yield the following equations:

$$\frac{du_n}{dt} = \alpha k_n[u_{n-1}^2 - 2u_{n+1}u_n - 2^{1/3}C(u_{n-1}u_n - 2u_{n+1}^2)] - \nu k_n^2 u_n$$
$$- \frac{1}{\rho_m u_n}[(\Delta E_E)_n + (\Delta E_T)_n + (\Delta E_S)_n + (\Delta E_B)_n + (\Delta E_D)_n] \quad [61]$$

$$\frac{d\beta_n(t)}{dt} = \beta_n v_n k_n[\log\beta_n - \log(\beta_{n-1}2^{D-3})] - (\Delta\beta)_n \quad [62]$$

$$\frac{\partial N_n(v,t)}{\partial t} = -g_n(v)N_n(v,t)\int_v^{(v_{\max})_n} v_n(v')\beta_n(v,v')g_n(v')N_n(v',t)\,dv'$$
$$+\int_0^{v/2} \lambda_n(v-v',v')h_n(v-v',v')N_n(v-v',t)N_n(v',t)\,dv'$$
$$-N_n(v,t)\int_0^{(v_{\max})_n - v} \lambda_n(v,v')h_n(v,v')N_n(v',t)\,dv' + F_s(v) + F_T(v) + F_D(v). \quad [63]$$

The drop eddy functions $(\Delta E_i)_n$, $(\Delta\beta)_n$, and $F_i(v)$ (where i represents an interaction type) are given by [29], [30] and [50]–[60].

Equations [61]–[63] must be integrated from a specified initial state [19] until the energy spectrum, the intermittency spectrum and the drop population reach steady state. The overall drop population can be constructed by combining the number densities of drops in all eddy sizes. In the companion paper (Jairazbhoy & Tavlarides 1995), a numerical scheme devised to integrate these equations is presented and the results analyzed.

3. CONCLUSIONS

In this paper, it has been demonstrated that the cascade model can be used as a framework within which drop–eddy interactions in two-phase flow can be modeled. The energy budget of the turbulent cascade is extended to include drop–eddy and drop–drop terms. An eddy intermittency equation is formulated to account for the destructive effects of drop–eddy collisions. A population balance equation is written for each eddy size. This balance accounts for drop breakage, drop coalescence and all drop–eddy events which change the number densities of drops in eddies of various sizes. The turbulence itself is viewed as a conglomeration of successively contained eddies, and the interactions are initiated primarily by drop–eddy collisions. The effect on the energy budget, eddy intermittency and drop populations is predicted by examination of the path of the drop, assuming a simple flow field.

The advantage of the present formulation is that it throws light on the whole energy spectrum. The mutual interactions between eddies and drops are likely to be much different if the eddies are large rather than small. In other words, large eddies tend to convect drops, while small eddies tend to shear them. Such effects are impossible to describe unless the entire energy spectrum is brought into focus. In addition, the role of the dispersed phase in two-phase turbulence is treated in a fundamental manner, inasmuch as an attempt is made to visualize the possible outcomes of drop–eddy encounters. The disadvantages, on the other hand, lie primarily in the solution of the large set of partial integro-differential equations that result, especially in the description of convective two-phase flows of practical interest. In the following paper (Jairazbhoy & Tavlarides 1995), a numerical scheme to effect such a solution, and the accompanying difficulties, are discussed. It is also suggested that, with the aid of fast computers, the cascade model may be adapted for non-homogeneous two-phase turbulent flows.

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